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The Continuity in G-metric Spaces Via Gβ- open Set

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ABSTRACT

In this paper we introduce and investigate weak form of G-continuous functions in G-metric spaces, namely G^{β}-continuous functions, via G^{β}-open sets. We give the notions of contra G^{β}-continuous functions, almost contra G^{β}-continuous functions, weakly G^{β}-continuous functions and slightly G^{β}-continuous functions.

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Keywords

continuous function; Metric spaces.

1. INTRODUCTION

In 2006 Mustafa and Sims , [2], introduced a new approach to generalized metric spaces, called G-metric space, and also introduced the notion of G-continuous functions. In 2021, [3], we introduced the concept of G^{β}-open sets by utilizing the open balls in G-metric spaces.

DEFINITION 1.1. [2] Let X be a nonempty set and R be the set of real numbers. A function $G : X \times X \times X \to R$ is called a G-metric function on X if it satisfies the following:

(1) $G(x, x, y) > 0$ for all $x \neq y \in X$;
(2) $G(x, y, z) = 0$ if and only if $x = y = z$;
(3) $G(x, x, y) \leq G(x, y, z)$ for every $x, y, z \in X$ with $y \neq z$;
(4) $G(x, y, z) = G(p(x, y, z))$ for every $x, y, z \in X$and for any permutation p of x, y, z;
(5) $G(x, y, z) \leq G(x, u, u)+G(u, y, z)$ for every $x, y, z, u \in X$.

If G is a G-metric function on X, then the pair (X, G) is called a G-metric space.

Let (X, G) be a G-metric space, $x \in X$ and $A \subseteq X$. The open ball with center x and radius ϵ in metric space (X, G) is denoted by $B_G(x, \epsilon)$ and defined by

$$B_G(x, \epsilon) = \{y \in X | d(x, y, y) < \epsilon\}.$$

The closed ball with center x and radius ϵ in G-metric space (X, G) is denoted by $C_G(x, \epsilon)$and defined by

$$C_G(x, \epsilon) = \{y \in X | d(x, y, y) \leq \epsilon\}.$$

The set A is called an open set in G-metric space (X, G) if for every $x \in A$, there is $\epsilon > 0$ such that $B_G(x, \epsilon) \subseteq A$. The set A is called closed set in metric space (X, G) if $X - A$ is an open set in G-metric space (X, G).

DEFINITION 1.2. [2] Let (X, G) and (X', G') be two G-metric spaces. The function $f : (X, G) \to (X', G')$ is called G-continuous at a point $a \in X$ if given $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$ and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \epsilon$. A function f is G-continuous if it is G-continuous at all points $a \in X$.

THEOREM 1.3. [2] Let (X, G) and (X', G') be two G-metric spaces. A function $f : (X, G) \to (X', G')$ is G-continuous if and only if $f^{-1}(H)$ is an open set in (X, G) for every open set H in (X', G').

Let (X, G) be a G-metric space and $A \subseteq X$. A point $x \in X$ is called a G-point of A in G-metric space (X, G), [3] if there is $\delta > 0$ such that for every $y \in B_G(x, \delta)$,

$$B_G(y, \epsilon) \cap G \neq \emptyset \quad \forall \epsilon > 0.$$

$G^{\beta}(A)$ denotes the set of all G^{β}-points of A in G-metric space (X, G)

DEFINITION 1.4. [3] Let (X, G) be a G-metric space. A subset $A \subseteq X$ is called a G^{β}-open set in G-metric space (X, G) if for every $x \in A$,

$$B_G(x, \epsilon) \cap \ G^{\beta}(A) \neq \emptyset \quad \forall \epsilon > 0.$$

A subset $A \in X$ is called a G^{β}-closed set in G-metric space (X, G) if $X - A$ is a G^{β}-open set in G-metric space (X, G).

THEOREM 1.5. *Every open set is a G^{β}-open set.*

This paper is organized as follows. Section 2 introduces a class of G^{β}-continuous functions in G-metric space. Section 3 gives the notions of contra G^{β}-continuous functions, almost contra G^{β}-continuous functions, weakly G^{β}-continuous functions and slightly G^{β}-continuous functions.

2. G^{β}-CONTINUOUS FUNCTION

DEFINITION 2.1. Let (X, G) and (X', G') be two G-metric spaces. A function $f : (X, G) \to (X', G')$ of a G-metric space (X, G) into a G-metric space (X', G') is called G^{β}-continuous function if $f^{-1}(U)$ is a G^β-open set in (X,G) for every open set U in (X',G').

THEOREM 2.2. Let (X,G) and (X',G') be two G-metric spaces. A function $f:(X,G)\to(X',G')$ of a G-metric space (X,G) into a G-metric space (X',G') is G^β-continuous if and only if $f^{-1}(F)$ is a G^β-closed set in (X,G) for every closed set F in (X',G').

PROOF. Let $f:(X,G)\to(X',G')$ be a G^β-continuous and F be any closed set in (X',G'). Then $f^{-1}(X'-F)=X-f^{-1}(F)$ is a G^β-open set in (X,G), that is, $f^{-1}(F)$ is G^β-closed set in (X,G). Conversely, suppose that $f^{-1}(F)$ is a G^β-closed set in (X,G) for every closed set F in (X',G'). Let U be any open set in (X',G'). Then by the hypothesis, $f^{-1}(X'-U)=X-f^{-1}(U)$ is is a G^β-closed set in (X,G), that is, $f^{-1}(U)$ is a G^β-open set in (X,G). Hence f is a G^β-continuous. □

THEOREM 2.3. Every G-continuous function is G^β-continuous function.

PROOF. Let $f:(X,G)\to(X',G')$ be a G-continuous function and U be any open set in (X',G'). Then $f^{-1}(U)$ is an open set in (X,G) and by Theorem (1.5), $f^{-1}(U)$ is a G^β-open set in (X,G). That is, f is a G^β-continuous function. □

The proof of the following lemma is similar for the proof of Theorem (2.2).

LEMMA 2.4. A function $f:(X,G)\to(X',G')$ of a G-metric space (X,G) into a G-metric space (X',G') is G^β-continuous if and only if $f^{-1}(F)$ is a G^β-closed set in (X,G) for every closed set F in (X',G').

Let (X,G) be a G-metric space and $A\subseteq X$. The closure operator of A is denoted by $Cl^X(A)$ and defined by

$$Cl^X(A)=\cap\{H\subseteq X: A\subseteq H \text{ and } H \text{ is closed set}\}.$$

The interior functor of A is denoted by $Int^X(A)$ and defined by

$$Int^X(A)=\cup\{H\subseteq X: H\subseteq A \text{ and } H \text{ is open set}\}.$$

The G-closure operator of A is denoted by $Cl^\beta_G(A)$ and defined by

$$Cl^\beta_G(A)=\cap\{H\subseteq X: A\subseteq H \text{ and } H \text{ is } G^\beta\text{-closed set}\}.$$

The G-interior functor of A is denoted by $Int^\beta_G(A)$ and defined by

$$Int^\beta_G(A)=\cup\{H\subseteq X: H\subseteq A \text{ and } H \text{ is } G^\beta\text{-open set}\}.$$

THEOREM 2.5. A function $f:(X,G)\to(X',G')$ of a G-metric space (X,G) into a G-metric space (X',G') is G^β-continuous if and only if $f[Cl^\beta_G(A)]\subseteq Cl^{X'}(f(A))$ for all $A\subseteq X$.

PROOF. Let f be a G^β-continuous and A be any subset of (X,G). Then $Cl^{X'}(f(A))$ is a closed set in (X',G'). Since f is a G^β-continuous then by Lemma (2.4), $f^{-1}[Cl^{X'}(f(A))]$ is a G^β-closed set in (X,G). That is,

$$Cl^\beta_G\big[f^{-1}[Cl^{X'}(f(A))]\big]=f^{-1}[Cl^{X'}(f(A))].$$

Since $f(A)\subseteq Cl^{X'}(f(A))$ then $A\subseteq f^{-1}[Cl^{X'}(f(A))]$. This implies,

$$Cl^\beta_G(A)\subseteq Cl^\beta_G\big[f^{-1}[Cl^{X'}(f(A))]\big]=f^{-1}[Cl^{X'}(f(A))].$$

Hence $f[Cl^\beta_G(A)]\subseteq Cl^{X'}(f(A))$.
Conversely, let H be any closed set in (X',G'), that is, $Cl^{X'}(H)=H$. Since $f^{-1}(H)\subseteq X$. Then by the hypothesis,

$$f\big[Cl^\beta_G[f^{-1}(H)]\big]\subseteq Cl^{X'}[f(f^{-1}(H))]\subseteq Cl^{X'}(H)=H.$$

This implies, $Cl^\beta_G[f^{-1}(H)]\subseteq f^{-1}(H)$. Hence $Cl^\beta_G[f^{-1}(H)]=f^{-1}(H)$, that is, $f^{-1}(H)$ is a G^β-closed set in (X,G). Hence by Lemma (2.4), f is a G^β-continuous. □

THEOREM 2.6. A function $f:(X,G)\to(X',G')$ of a G-metric space (X,G) into a G-metric space (X',G') is G^β-continuous if and only if $Cl^\beta_G(f^{-1}(B))\subseteq f^{-1}(Cl^{X'}(B))$ for all $B\subseteq X'$.

PROOF. Let f be a G^β-continuous and B be any subset of (X',G'). Then $Cl^{X'}(B)$ is a closed set in (X',G'). Since f is a G^β-continuous then by Lemma (2.4), $f^{-1}[Cl^{X'}(B)]$ is a G^β-closed set in (X,G). That is,

$$Cl^\beta_G\big[f^{-1}[Cl^{X'}(B)]\big]=f^{-1}[Cl^{X'}(B)].$$

Since $B\subseteq Cl^{X'}(B)$ then $f^{-1}(B)\subseteq f^{-1}[Cl^{X'}(B)]$. This implies,

$$Cl^\beta_G(f^{-1}(B))\subseteq Cl^\beta_G\big[f^{-1}[Cl^{X'}(B)]\big]=f^{-1}[Cl^{X'}(B)].$$

Hence $Cl^\beta_G(f^{-1}(B))\subseteq f^{-1}[Cl^{X'}(B)]$.
Conversely, let H be any closed set in (X',G'), that is, $Cl^{X'}(H)=H$. Since $H\subseteq X'$. Then by the hypothesis,

$$Cl^\beta_G(f^{-1}(H))\subseteq f^{-1}(Cl^{X'}(H))=f^{-1}(H).$$

This implies, $Cl^\beta_G[f^{-1}(H)]\subseteq f^{-1}(H)$. Hence $Cl^\beta_G[f^{-1}(H)]=f^{-1}(H)$, that is, $f^{-1}(H)$ is a G^β-closed set in (X,G). Hence by Lemma (2.4), f is a G^β-continuous. □

THEOREM 2.7. A function $f:(X,G)\to(X',G')$ of a G-metric space (X,G) into a G-metric space (X',G') is G^β-continuous if and only if $f^{-1}(Int^{X'}(B))\subseteq Int^\beta_G[f^{-1}(B)]$ for all $B\subseteq X'$.

PROOF. Let f be a G^β-continuous and B be any subset of (X',G'). Then $Int^{X'}(B)$ is an open set in (X',G'). Since f is a G^β-continuous then $f^{-1}[Int^{X'}(B)]$ is a G^β-open set in (X,G). That is,

$$Int^\beta_G\big[f^{-1}[Int^{X'}(B)]\big]=f^{-1}[Int^{X'}(B)].$$

Since $Int^{X'}(B)\subseteq B$ then $f^{-1}[Int^{X'}(B)]\subseteq f^{-1}(B)$. This implies,

$$f^{-1}[Int^{X'}(B)]=Int^\beta_G\big[f^{-1}[Int^{X'}(B)]\big]\subseteq Int^\beta_G(f^{-1}(B)).$$

Hence $f^{-1}(Int^{X'}(B))\subseteq Int^\beta_G[f^{-1}(B)]$.
Conversely, let U be any open set in (X',G'), that is, $Int^{X'}(U)=U$. Since $U\subseteq X'$. Then by the hypothesis,

$$f^{-1}(U)=f^{-1}(Int^{X'}(U))\subseteq Int^\beta_G[f^{-1}(U)].$$

This implies, $f^{-1}(U)\subseteq Int^\beta_G[f^{-1}(U)]$. Hence $f^{-1}(U)=Int^\beta_G[f^{-1}(U)]$, that is, $f^{-1}(U)$ is a G^β-open set in (X,G). Hence f is a G^β-continuous. □

3. CONTRA G^{β}-CONTINUOUS FUNCTIONS

DEFINITION 3.1. A function $f : (X, G) \to (X', G')$ of a G-metric space (X, G) into a G-metric space (X', G') is called contra G^{β}-continuous function if $f^{-1}(V)$ is a G^{β}-closed set in (X, G) for every open set V in (X', G').

THEOREM 3.2. A function $f : (X, G) \to (X', G')$ is contra G^{β}-continuous if and only if $f^{-1}(F)$ is a G^{β}-open set in (X, G) for every closed set F in (X', G').

THEOREM 3.3. A function $f : (X, G) \to (X', G')$ is contra G^{β}-continuous if and only if for each $x \in X$ and each closed set F in (X', G') containing $f(x)$, there is a G^{β}-open set U in (X, G) containing x such that $f(U) \subseteq F$.

PROOF. Suppose that $f : (X, G) \to (X', G')$ is contra G^{β}-continuous. Let $x \in X$ and F be a closed set in (X', G') containing $f(x)$. Then by the last theorem, $U = f^{-1}(F)$ is a G^{β}-open set in (X, G). Since $f(x) \in F$ then $x \in f^{-1}(F) = U$ and $f(U) = f(f^{-1}(F)) \subseteq F$.
Conversely, Let F be a closed set in (X', G'). For each $x \in f^{-1}(F)$, $f(x) \in F$. Then by the hypothesis, there is a G^{β}-open set U_x in (X, G) containing x such that $f(U_x) \subseteq F$. Therefore, we obtain

$$f^{-1}(F) = \cup\{U_x : x \in f^{-1}(F)\}.$$

Then $f^{-1}(F)$ is a G^{β}-open set in (X, G). Hence by the last theorem, f is a contra G^{β}-continuous. □

DEFINITION 3.4. A function $f : (X, G) \to (X', G')$ of a G-metric space (X, G) into a G-metric space (X', G') is called almost G^{β}-continuous if for each $x \in X$ and each open set V in (X', G') containing $f(x)$, there is a G^{β}-open set U in (X, G) containing x such that $f(U) \subseteq Int^{\beta}_{G'}[Cl^{X'}(V)]$.

A function $f : (X, G) \to (X', G')$ of a G-metric space (X, G) into a G-metric space (X', G') is called G^{β}-open function if $f(V)$ is a G^{β}-open set in (X', G') for every G^{β}-open set V in (X, G).

THEOREM 3.5. If a function $f : (X, G) \to (X', G')$ is a G^{β}-open function and contra G^{β}-continuous then f is an almost G^{β}-continuous.

PROOF. Let $x \in X$ be any point in (X, G) and V be any open set in (X', G') containing $f(x)$. Since f ia contra G^{β}-continuous and $Cl^{X'}(V)$ be a closed set in (X', G') containing $f(x)$ then by Theorem (3.3), there is a G^{β}-open set U in (X, G) containing x such that $f(U) \subseteq Cl^{X'}(V)$. Since f is a G^{β}-open function and U is a G^{β}-open set in (X, G) then $f(U)$ is a G^{β}-open set in (X', G') and

$$f(U) = Int^{\beta}_{G'}[f(U)] \subseteq Int^{\beta}_{G'}[Cl^{X'}(f(U))] \subseteq Int^{\beta}_{G'}[Cl^{X'}(V)].$$

This shows that f is an almost G^{β}-continuous. □

DEFINITION 3.6. A function $f : (X, G) \to (X', G')$ of a G-metric space (X, G) into a G-metric space (X', G') is called weakly G^{β}-continuous function if for each $x \in X$ and each open set V in (X', G') containing $f(x)$, there is a G^{β}-open set U in (X, G) containing x such that $f(U) \subseteq Cl^{X'}(V)$.

It is clear that every a G^{β}-continuous function is a weakly G^{β}-continuous function.

A subset of G-metric space is called a clopen set if it is both open and closed set, similar for G^{β}-clopen set.

DEFINITION 3.7. A function $f : (X, G) \to (X', G')$ of a G-metric space (X, G) into a G-metric space (X', G') is called slightly G^{β}-continuous function if for each $x \in X$ and each clopen set U in (X', G') containing $f(x)$, there exists G^{β}-open set V in (X, G) containing x such that $f(V) \subseteq U$.

THEOREM 3.8. Let $f : (X, G) \to (X', G')$ be a function of a G-metric space (X, G) into a G-metric space (X', G'). Then the following are equivalent:

(1) f is slightly G^{β}-continuous.
(2) $f^{-1}(U)$ is a G^{β}-open set in (X, G) for every clopen set U in (X', G').
(3) $f^{-1}(U)$ is a G^{β}-closed set in (X, G) for every clopen set U in (X', G').
(4) $f^{-1}(U)$ is a G^{β}-cloopen set in (X, G) for every clopen set U in (X', G').

PROOF. $1 \Rightarrow 2$: Let U be a clopen set in (X', G'). For each $x \in f^{-1}(U)$, $f(x) \in U$. Since f is slightly G^{β}-continuous then there exists G^{β}-open set V_x in (X, G) containing x such that $f(V_x) \subseteq U$. This implies, $x \in V_x \subseteq f^{-1}(U)$. Hence

$$f^{-1}(U) = \cup\{V_x : x \in f^{-1}(U)\}.$$

That is, $f^{-1}(U)$ is a G^{β}-open set in (X, G).
$2 \Rightarrow 3$: Let U be a clopen set in (X', G'). Then $X' - U$ is a clopen set in (X', G'). By the hypothesis, $X - f^{-1}(U) = f^{-1}(X' - U)$ is a G^{β}-open set in (X, G). That is, $f^{-1}(U)$ is a G^{β}-closed set in (X, G).
$3 \Rightarrow 4$: It is easy from the previous.
$4 \Rightarrow 1$: Let $x \in X$ be any point in (X, G) and U be a clopen set in (X', G') containing $f(x)$. By the hypothesis, $f^{-1}(U)$ is a G^{β}-cloopen set in (X, G). Then $V = f^{-1}(U)$ is a G^{β}-open set V in (X, G) containing x such that $f(V) \subseteq U$. That is, f is slightly G^{β}-continuous. □

THEOREM 3.9. Every weakly G^{β}-continuous is slightly G^{β}-continuous .

PROOF. Let $f : (X, G) \to (X', G')$ be a weakly G^{β}-continuous function. Let $x \in X$ be any point in (X, G) and U be any clopen set in (X', G') containing $f(x)$. Then U is an open set in (X', G') containing $f(x)$. Then there is a G^{β}-open set V in (X, G) containing x such that $f(V) \subseteq Cl(U) = U$. Hence f is slightly G^{β}-continuous. □

4. REFERENCES

[1] Z.Mustafa and B.Sims, Some remarks concerning D-metric spaces Proceedings of the International Conferences on Fixed Point Theory and Applications, Valencia (Spain), (2003), 189-198.

[2] Z.Mustafa and B.Sims, A new approach to generalized metric spaces, Journal of Non-linear and Convex Analysis, 7, (2006), 289-297.

[3] A. Mubarak and A. Saif, On G^{β}-property in G-metric spaces, Asian Journal of Advanced Research and Reports (in press).